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LETTER TO THE EDITOR

A new linear Dirac-like spin-$\frac{3}{2}$ wave equation using Clifford algebra

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Abstract. We derive a new linear Dirac-like wave equation for spin-$\frac{3}{2}$ employing four of the seven irreducible eight-dimensional matrices obeying the Clifford algebra C_7 with the wavefunction having the needed eight components only. Though this wave equation is not manifestly covariant and the wavefunction employed is not locally covariant, it is relativistically invariant and by its very derivation is connected to the Weaver, Hammer and Good (WHG) formalism for spin-$\frac{3}{2}$ by a chain of transformations which can be arbitrarily chosen by us to be either unitary or non-unitary.

The purpose of this letter is to present an interesting derivation of a Dirac-like linear spin-$\frac{3}{2}$ wave equation starting from the recently discussed (Weaver *et al* 1964, Mathews 1966a, b, 1967a, b, Seetharaman *et al* 1971, Jayaraman 1973a, b, 1975) Schrödinger types of wave equations describing massive particles of arbitrary spin specialized to the case of spin-$\frac{3}{2}$.

In the WHG formalism of particles of spin-$\frac{3}{2}$ and mass m, the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\psi = H\psi, \qquad \psi = \begin{pmatrix} \psi^{(0,\frac{3}{2})} \\ \psi^{(\frac{3}{2},0)} \end{pmatrix}, \tag{1}$$

with ψ transforming locally according to the 8-dimensional representation $D(0,\frac{3}{2})\oplus D(\frac{3}{2},0)$ of the homogeneous Lorentz group, is invariant under the operations of the Poincaré group whose generators in the space of wavefunctions ψ are (Mathews 1966a)

$$P_0 = p_0 \equiv -\mathrm{i}\frac{\partial}{\partial t} = -H, \tag{2a}$$

$$\boldsymbol{P} = \boldsymbol{p} \equiv -\mathrm{i}\boldsymbol{\nabla}, \tag{2b}$$

$$\boldsymbol{J} = \boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{S}, \qquad \boldsymbol{S} = \begin{pmatrix} \boldsymbol{s} & 0 \\ 0 & \boldsymbol{s} \end{pmatrix}, \tag{2c}$$

$$\boldsymbol{K} = t\boldsymbol{p} + \boldsymbol{x}p_0 + \mathrm{i}\boldsymbol{\lambda}, \qquad \boldsymbol{\lambda} = \rho_3\boldsymbol{S} = \begin{pmatrix} \boldsymbol{s} & 0 \\ 0 & -\boldsymbol{s} \end{pmatrix}. \tag{2d}$$

Here the matrices $(s_1, s_2, s_3) = \boldsymbol{s}$ are a 4-dimensional representation of the angular momentum operator for spin-$\frac{3}{2}$ and have their explicit forms

$$s_1 = \frac{\sqrt{3}}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_- \\ \sigma_+ & 0 \end{pmatrix}, \tag{3a}$$

$$s_2 = \frac{\sqrt{3}}{2}\begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} + \begin{pmatrix} 0 & -\mathrm{i}\sigma_- \\ \mathrm{i}\sigma_+ & 0 \end{pmatrix}, \tag{3b}$$

$$s_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{3c}$$

with $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2)$. In (2) and (3), ρ_i and $\sigma_i (i = 1, 2, 3)$ are respectively the 8-dimensional and 2-dimensional Pauli matrices. The Hamiltonian H of equation (1) has the explicit form (Weaver *et al* 1964, Mathews 1966a)

$$H = E\left(\sum_{\nu=1/2}^{3/2} \tanh(2\nu\theta) C_\nu + \rho_1 \sum_{\nu=1/2}^{3/2} \operatorname{sech}(2\nu\theta) B_\nu \right). \tag{4}$$

In (4),

$$E = m \cosh\theta, \qquad p = m \sinh\theta \tag{5}$$

and B_ν and C_ν are the combinations

$$B_\nu = \Lambda_\nu + \Lambda_{-\nu}, \qquad C_\nu = \Lambda_\nu - \Lambda_{-\nu} \tag{6a}$$

of the projection operators Λ_ν to the eigenvalue ν of $\lambda_p = \boldsymbol{\lambda}.\boldsymbol{p}/p$ and satisfy

$$B_\mu B_\nu = C_\mu C_\nu = B_\mu \delta_{\mu\nu}, \qquad B_\mu C_\nu = C_\mu \delta_{\mu\nu}. \tag{6b}$$

Obviously H of (4) is not linear in $\boldsymbol{p}$.

We shall now proceed to prove our assertion that equation (1) employing the Hamiltonian (4) can be recast in a Dirac-like form linear in $\boldsymbol{p}$. To accomplish this one passes first from the ψ representation defined by equations (1)–(2) to Foldy's (1956) canonical representation ϕ by means of the non-unitary transformation R (Mathews 1966b)

$$\psi \to \phi = R\psi, \tag{7a}$$

$$R = \frac{1}{\sqrt{2}}(\rho_1 + \rho_3)\left(\frac{E}{m}\right)^{1/2} \sum_{\nu=1/2}^{3/2} [\cosh(\nu\theta) B_\nu + \rho_1 \sinh(\nu\theta) C_\nu] \operatorname{sech}(2\nu\theta), \tag{7b}$$

$$R^{-1} = \left(\frac{m}{E}\right)^{1/2} \sum_{\nu=1/2}^{3/2} [\cosh(\nu\theta) B_\nu - \rho_1 \sinh(\nu\theta) C_\nu] \frac{1}{\sqrt{2}}(\rho_1 + \rho_3). \tag{7c}$$

The Poincaré generators (2a)–(2d) in the ψ representation are transformed in the ϕ representation into

$$P_{0\phi} = p_0 \equiv -\mathrm{i}\frac{\partial}{\partial t} = -\mathscr{H}_\phi = -\rho_3 E, \tag{8a}$$

$$\boldsymbol{P}_\phi = \boldsymbol{p} \equiv -\mathrm{i}\boldsymbol{\nabla}, \tag{8b}$$

$$\boldsymbol{J}_\phi = \boldsymbol{x} \times \boldsymbol{p} + \boldsymbol{S}, \tag{8c}$$

$$\boldsymbol{K}_\phi = t\boldsymbol{p} - \tfrac{1}{2}\rho_3(\boldsymbol{x}E + E\boldsymbol{x}) + \rho_3[\boldsymbol{S} \times \boldsymbol{p}/(E+m)]. \tag{8d}$$

In (8a) the Pauli matrix ρ_3 can be identified with the 8×8 matrix derived by the direct product

$$\rho_3 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = \Gamma_0 = \sigma_3 \otimes I \tag{9a}$$

where I is a four-dimensional unit matrix. Now, Γ_0 together with

$$\Gamma_a = 2i\sigma_2 \otimes l_a, \tag{9b}$$

$$\Gamma_{a+3} = 2i\sigma_1 \otimes \tau_a, \qquad (a = 1,2,3), \tag{9c}$$

constitute an 8-dimensional irreducible representation of the Clifford algebra C_7 (Fushchich 1974, Miller 1972)

$$[\gamma_i, \gamma_j]_+ = 2\delta_{ij}, \qquad (i, j = 0, 1, \ldots, 6), \tag{10a}$$

$$\gamma_0 = \Gamma_0, \qquad \gamma_a = \Gamma_0 \Gamma_a, \qquad \gamma_{a+3} = \Gamma_0 \Gamma_{a+3}, \qquad (a = 1,2,3), \tag{10b}$$

$$\gamma_i^\dagger = \gamma_i. \tag{10c}$$

The l_a and τ_a are the following 4×4 matrices employed by Fushchich (1974) in his construction of a Poincaré invariant wave equation for two spin-$\frac{1}{2}$ particles of arbitrary mass:

$$\begin{gathered}
l_1 = \frac{1}{2}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad l_2 = \frac{1}{2}\begin{pmatrix} 0 & \frac{1}{2}(1+i)+\frac{1}{2}(1-i)\sigma_3 \\ \frac{1}{2}(1-i)+\frac{1}{2}(1+i)\sigma_3 & 0 \end{pmatrix}, \\
l_3 = \frac{1}{2}\begin{pmatrix} 0 & \sigma_+ - i\sigma_- \\ i\sigma_+ + \sigma_- & 0 \end{pmatrix}, \\
\tau_1 = \frac{1}{2}\begin{pmatrix} -\sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad \tau_2 = \frac{1}{2}\begin{pmatrix} 0 & -\frac{1}{2}(1-i)-\frac{1}{2}(1+i)\sigma_3 \\ -\frac{1}{2}(1+i)-\frac{1}{2}(1-i)\sigma_3 & 0 \end{pmatrix}, \\
\tau_3 = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_+ - i\sigma_- \\ i\sigma_+ - \sigma_- & 0 \end{pmatrix}.
\end{gathered} \tag{11}$$

It is now a simple matter to define the new χ representation by the unitary transformation

$$\phi \to \chi = U\phi, \tag{12}$$

$$U = \exp\left(-\theta_1 \Gamma_0 \frac{\boldsymbol{\gamma} \cdot \boldsymbol{p}}{p}\right), \tag{13a}$$

$$U^{-1} = \exp\left(\theta_1 \Gamma_0 \frac{\boldsymbol{\gamma} \cdot \boldsymbol{p}}{p}\right) = U^\dagger \tag{13b}$$

with γ_a given by (10*b*) and

$$\cos\theta_1 = \frac{E+m}{[2E(E+m)]^{1/2}}, \qquad \sin\theta_1 = \frac{p}{[2E(E+m)]^{1/2}}. \tag{14}$$

With (14), U and U^{-1} take the explicit forms

$$U = (E + m - \Gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{p})/[2E(E+m)]^{1/2}, \tag{15a}$$

$$U^{-1} = U^\dagger = (E + m + \Gamma_0 \boldsymbol{\gamma} \cdot \boldsymbol{p})/[2E(E+m)]^{1/2} \tag{15b}$$

which very much resemble the spin-$\frac{1}{2}$ Foldy–Wouthuysen (1950) transformation operators with the difference that the Γ here are eight-dimensional.

It is straightforward to check that

$$\mathscr{H}_\chi = U\mathscr{H}_\phi U^\dagger = \Gamma_0\boldsymbol{\Gamma}\cdot\boldsymbol{p} + \Gamma_0 m \tag{16}$$

so that the wave equation in the χ representation becomes

$$\mathrm{i}(\partial\chi/\partial t) = (\Gamma_0\boldsymbol{\Gamma}\cdot\boldsymbol{p} + \Gamma_0 m)\chi \tag{17}$$

which is the new Dirac-like equation for spin-$\frac{3}{2}$ linear in $\boldsymbol{p}$.

It may be noted that the wavefunction χ in (17) is not locally covariant. It is well known that Foldy's wavefunction (Foldy 1956) ϕ is also not locally covariant. However equation (17) is relativistically invariant in the sense that

$$[\mathrm{i}(\partial/\partial t) - \mathscr{H}_\chi, Q_\chi]\chi = 0 \tag{18}$$

where Q_χ are the transformed Poincaré generators

$$P_{0\chi} = -\mathscr{H}_\chi = -(\Gamma_0\Gamma_a p_a + \Gamma_0 m), \qquad (a = 1,2,3), \tag{19a}$$

$$\boldsymbol{P}_\chi = -\mathrm{i}\boldsymbol{\nabla}, \tag{19b}$$

$$\boldsymbol{J}_\chi = U(\boldsymbol{x}\times\boldsymbol{p} + \boldsymbol{S})U^\dagger, \tag{19c}$$

$$\boldsymbol{K}_\chi = U\{t\boldsymbol{p} - \tfrac{1}{2}\rho_3(\boldsymbol{x}E + E\boldsymbol{x}) + [\rho_3\boldsymbol{S}\times\boldsymbol{p}/(E+m)]\}U^\dagger \tag{19d}$$

with U defined by (15).

A relativistically invariant scalar product between any two solutions of (17) takes the form

$$\int \chi_1^\dagger\chi_2\mathrm{d}^3x = \int \phi_1^\dagger U^\dagger U\phi_2\mathrm{d}^3x = \int \phi_1^\dagger\phi_2\mathrm{d}^3x \tag{20}$$

which follows from the form $\int\phi_1^\dagger\phi_2\mathrm{d}^3x$ for the scalar product (Mathews 1966b) in Foldy's canonical representation and the fact that χ is unitarily related to ϕ by (12).

For the WHG representation ψ, the scalar product has the form (Mathews 1966b)

$$\int \psi_1^\dagger M\psi_2\mathrm{d}^3x = \int \psi_1^\dagger R^\dagger R\psi_2\mathrm{d}^3x = \int \phi_1^\dagger\phi_2\mathrm{d}^3x \tag{21}$$

where the metric operator M is

$$M = (E/m)\sum_{\nu=1/2}^{3/2}\mathrm{sech}(2\nu\theta)B_\nu. \tag{22}$$

It may be noted that the transformation operator R is non-unitary though U is unitary. As was observed in a recent paper (Jayaraman 1976) one can also construct unitary transformations

$$\psi \to \phi_I = R_I\psi, \tag{23}$$

$$\phi_I \to \chi_I = U\phi_I \tag{24}$$

where

$$R_I = \frac{1}{\sqrt{2}}(\rho_2 + \rho_3)\left(1 + \rho_2\frac{H}{E}\right), \tag{25a}$$

$$R_I^{-1} = R_I^\dagger = \frac{1}{\sqrt{2}}\left(1 - \rho_2\frac{H}{E}\right)(\rho_2 + \rho_3) \tag{25b}$$

and U is the same transformation operator defined in equation (15). In the ϕ_I representation the Poincaré generators $P_{0\phi_I}$, $\boldsymbol{P}_{\phi_I}$ and $\boldsymbol{J}_{\phi_I}$ have the same form as $P_{0\phi}$, $\boldsymbol{P}_\phi$ and $\boldsymbol{J}_\phi$ of equations (8*a*)–(8*c*) (Jayaraman 1976) while

$$\boldsymbol{K}_{\phi_I} = t\boldsymbol{p} - \boldsymbol{x}\rho_3 E + \tfrac{1}{2}\rho_1 \boldsymbol{S}(\rho_2+\rho_3)H(\rho_2+\rho_3). \tag{26}$$

In the χ_I representation the Poincaré generators $P_{0\chi_I}$, $\boldsymbol{P}_{\chi_I}$ and $\boldsymbol{J}_{\chi_I}$ have the same expressions as in (19*a*)–(19*c*) while

$$\boldsymbol{K}_{\chi_I} = U[t\boldsymbol{p} - \boldsymbol{x}\rho_3 E + \tfrac{1}{2}\rho_1 \boldsymbol{S}(\rho_2+\rho_3)H(\rho_2+\rho_3)]U^\dagger. \tag{27}$$

Clearly the wave equation in the χ_I representation again takes the Dirac-like form

$$\mathrm{i}\frac{\partial \chi_I}{\partial t} = (\Gamma_0 \boldsymbol{\Gamma}.\boldsymbol{p} + \Gamma_0 m)\chi_I. \tag{28}$$

It is not difficult to see that the relativistically invariant scalar products in the ϕ_I and χ_I representations are respectively

$$\int \phi_{I1}^\dagger M \phi_{I2}\,\mathrm{d}^3x = \int \psi_1^\dagger M \psi_2\,\mathrm{d}^3x \tag{29}$$

and

$$\int \chi_{I1}^\dagger M_1 \chi_{I2}\,\mathrm{d}^3x \tag{30}$$

where M is defined by (22) and

$$M_1 = U^{-1}MU = U^\dagger MU. \tag{31}$$

The explicit evaluation of (19*c*)–(19*d*), (27) and (31) can be accomplished with the representation of the matrices $\boldsymbol{S} = 1 \otimes \boldsymbol{s}$ (1 being a 2×2 unit matrix) with $\boldsymbol{s}$ of equations (3*a*)–(3*c*) and the Γ of equations (9)–(11). The details of these calculations, and as well a discussion on the massless limit of equations (17) and (28) will be the subject matter for a future publication.

An explanation may be in order regarding the role of the matrices Γ_4, Γ_5 and Γ_6 of (9*c*) which do not find a place in the new wave equation. Their presence need not worry us any more than the fifth anticommuting matrix γ_5 which is a product

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \qquad (\gamma_5^\dagger = \gamma_5,\ \gamma_5^2 = 1,\ [\gamma_5, \gamma_i]_+ = 0,\ (i = 1,2,3,4)) \tag{32}$$

of the four anticommuting Dirac matrices in the spin-$\frac{1}{2}$ formalism and which does not find a place in the Dirac spin-$\frac{1}{2}$ equation. (In (32) the γ are 4-dimensional.) The presence of the additional anticommuting Γ matrices for spin-$\frac{3}{2}$ may even suggest some additional symmetry properties of the wave equation.

Finally it may be observed that since the Clifford algebra C_{2n+1} (n being an integer) has two inequivalent irreducible representations (Miller 1972) of dimensionality 2^n, the construction of a linear Dirac-like wave equation is feasible for such half-integer spins s which satisfy

$$2(2s+1) = 2^n, \qquad (n = 2,3,4,5,\ldots). \tag{33}$$

Obviously the solution of (33) is that

$$s = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, \tfrac{15}{2}, \ldots. \tag{34}$$

A fuller discussion for spins $s > \frac{3}{2}$ will be reported elsewhere.

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